Direct Methods for solving linear systems:

Linear systems of equations are associated with many problems in engineering and Science, as well as with applications of mathematics for social Sciences.

Direct techniques are considered to solve the linear system

$$
\begin{aligned}
&a_1,x_1 + a_{1.2}x_2 + \ldots a_{1n}x_n = b_1 \\
&a_2,x_1 + a_{22}x_2 + \ldots a_{2n}x_n = b_2 \\
&\vdots \\
&a_n,x_1 + a_{n2}x_2 + \ldots a_{nn}x_n = b_n
\end{aligned}
$$

for $x_1,\ldots,x_n$ given the a_{ij} for each $i, j = 1, 2, \ldots, n$

and b_i for each $i = 1, 2, \ldots n$.

Direct techniques are methods that give an answer in a fixed number of steps subject only to rounding errors.

Linear system of equations:

Examples The four equation's

$$x_1 + x_2 + 3x_4 = 4 \qquad (1)$$

$$2x_1 + x_2 - x_3 + x_4 = 1 \qquad (2)$$

$$3x_1 - x_2 - x_3 + 2x_4 = 3 \qquad (3)$$

$$-x_1 + 2x_2 + 3x_3 - x_4 = 4 \qquad (4)$$

x_1, x_2, x_3 x_4 x_1

will be solved for the unknowns and the

equation (2), (3) and (4) by performing

(2) – 2 (1), (3) -3 (1) and (4) + (1)

the resulting system is

$$x_1 + x_2 + 3x_4 = 4 \qquad (1`)$$

$$x_2 - x_3 - 5x_4 = -7 \qquad (2`)$$

$$-4x_2 - x_3 - 7x_4 = -15 \qquad (3`)$$

$$3x_2 + 3x_3 + 2x_4 = 8 \qquad (4`)$$

x_2

where the new equations are labled (1`), (2`), (3`) and (4`) in this system (2`) is used to eliminate from (3`) and (4`) by the operations

(3`) – 4(2`) and (4`) + 3(2`) resulting in the system

$$
\begin{aligned}
x_1 + x_2 + \quad 3x_4 &= 4 \\
-x_2 - x_3 - 5x_4 &= -7 \\
3x_3 + 13x_4 &= 13 \\
-13x_4 &= -13
\end{aligned}
$$

the system now in reduced form and can easily be solved for the unknowns by a back ward substitution process, noting that $x_4 = 1$

the solution is therefore $x_1 = -1,\ x_2 = 2,\ x_3 = 0$

and $x_4 = 1$

Gaussian Elimination:

Definition:

an n x m matrix is a rectangular array of elements with n rows and m columns in which not only is the value of an element important but also its position in the array.

$$A = \left(a_{ij}\right) = \left.\underbrace{\begin{bmatrix} a_{1.1} & a_{1.2} & a_{1.3} & \cdots & a_{1m} \\ a_{2.1} & a_{2.2} & a_{2.3} & \cdots & a_{2m} \\ \vdots & & & & \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nm} \end{bmatrix}}_{Columns}\right\} rows$$

An *(n+1) x n* matrix can be used to represent the linear system

$$
\begin{aligned}
&a_{1.1}x_1 + a_{1.2}x_2 + \ldots + a_{1.n}x_n = b_1\\
&a_{2.1}x_1 + a_{2.2}x_2 + \ldots + a_{2.n}x_n = b_2\\
&\vdots\\
&a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n = b_n
\end{aligned}
$$

by first constructing

$$
A = \begin{bmatrix} a_{1.1} & a_{1.2} & \ldots & a_{1n} \\ a_{2.1} & a_{2.2} & \ldots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{bmatrix} \; and \; b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}
$$

and then combining these matrices to form the augmented matrix:

$$[A,b] = \begin{bmatrix} a_{1.2} & a_{1.2} & \cdots & a_{1.n} & \vdots & b_1 \\ a_{2.1} & a_{2.2} & \cdots & a_{2.n} & \vdots & b_2 \\ \vdots & & & & \vdots & \\ a_{n1} & a_{n2} & & a_{nn} & \vdots & b_n \end{bmatrix}$$

where the broken line is used to separate the coefficients of the unknown from the values on the right hand side of the equations.

Now, repeating the operations involved in Example (1) in considering first the augmented matrix associated with the system

$$\begin{bmatrix} 1 & 1 & 0 & 3 & \vdots & 4 \\ 2 & 1 & -1 & 1 & \vdots & 1 \\ 3 & -1 & -1 & 2 & \vdots & -3 \\ -1 & 2 & 3 & -1 & \vdots & 4 \end{bmatrix} \quad *$$

performing the operations associated with

(2) – 2 (1), (3) -3 (1) and (4) + (1) is accomplished by manipulating the respective rows of the augmented matrix * which becomes the

matrix

$$\begin{bmatrix} 1 & 1 & 0 & 3 & \vdots & 4 \\ 2 & -1 & -1 & -5 & \vdots & -7 \\ 0 & -4 & -1 & -7 & \vdots & -15 \\ 0 & 3 & 3 & 2 & \vdots & 8 \end{bmatrix}$$

performing the final manipulation results in the augmented matrix

$$\begin{bmatrix} 1 & 1 & 0 & 3 & \vdots & 4 \\ 0 & -1 & -1 & -5 & \vdots & -7 \\ 0 & 0 & 3 & 13 & \vdots & 13 \\ 0 & 0 & 0 & -13 & \vdots & -13 \end{bmatrix}$$

this matrix can be transformed into its corresponding linear system and solutions for x_1, x_2, x_3 and x_4 obtained.

The procedure involved in this process is called Gaussian elimination with backward substitution.

<u>Gaussian Elimination:</u>

The general form applied to the linear system

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \ \dots \ + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \ \dots \ + a_{2n}x_n &= b_2 \\
&\vdots \\
a_{n1}x_1 + a_{n2}x_2 + \ \dots \ + a_{nn}x_n &= b_n
\end{aligned}
$$

$$A = [A, b] = \begin{bmatrix} a_{1.1} & a_{1.2} & \dots & a_{1n} & 1 & a_{1,n+1} \\ a_{2.1} & a_{2.2} & \dots & a_{2n} & 1 & a_{2,n+1} \\ a_{n1} & a_{n2} & \dots & a_{nn} & 1 & a_{n,n+1} \end{bmatrix}$$

the resulting matrix will be

$$\tilde{A} = \begin{bmatrix} a_{1.1} & a_{1.2} \dots a_{1n} & 1 & a_{1,n+1} \\ 0 & a_{2.2} \dots a_{2n} & 1 & a_{2,n+1} \\ 0 & 0 \quad\quad a_{nn} & 1 & a_{n,n+1} \end{bmatrix}$$

the back ward substitution can be performed solving the n^th equation for x_n gives

$$x_n = \frac{a_n, n+1}{a_{nn}}$$

and so on α_o, x_{n-1}

Example:

Solve the linear system using the elimination method:

$$\begin{aligned}
x_1 + x_2 + x_3 + x_4 &= 7 \\
x_1 + x_2 \qquad\quad + 2x_4 &= 8 \\
2x_1 + 2x_2 + 3x_3 \qquad &= 10 \\
-x_1 - x_2 - 2x_3 + 2x_4 &= 0
\end{aligned}$$

Solution:

$$\tilde{A} = \left[A, b \right] = \begin{bmatrix} 1 & 1 & 1 & 1 & \vdots & 7 \\ 1 & 1 & 0 & 2 & \vdots & 8 \\ 2 & 2 & 3 & 0 & \vdots & 10 \\ -1 & -1 & -2 & 2 & \vdots & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 & 1 & 1 & \vdots & 7 \\ 0 & 0 & -1 & 1 & \vdots & 1 \\ 0 & 0 & 1 & -2 & \vdots & -4 \\ 0 & 0 & -1 & 3 & \vdots & 7 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 1 & 1 & \vdots & 7 \\ 0 & 0 & -1 & 1 & \vdots & 1 \\ 0 & 0 & 1 & -2 & \vdots & 4 \\ 0 & 0 & 0 & 1 & \vdots & 3 \end{bmatrix}$$

performing backward substitution

$$x_4 = 3$$

$$x_3 - 2x_4 = -4 \qquad \therefore x_3 = 2$$

x_2 arbitrary and $x_1 = 2 - x_2$

there is no unique solution.

Example: Solve the linear system using the elimination method

$$\begin{aligned} x_1 - x_2 + 3x_3 &= 2 \\ 3x_1 - 3x_2 + x_3 &= -1 \\ x_1 + x_2 \qquad &= 3 \end{aligned}$$

Solution:

Row interchange necessary

$$\therefore\ \tilde{A} = [A, b] = \begin{bmatrix} 1 & -1 & 3 & \vdots & 2 \\ 1 & 1 & 0 & \vdots & 3 \\ 3 & -3 & 1 & \vdots & -1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -1 & 3 & \vdots & 2 \\ 0 & 2 & -3 & \vdots & 1 \\ 0 & 0 & -8 & \vdots & -7 \end{bmatrix}$$

$$x_3 = \forall_8 = 0.875$$

$$2x_2 - 3x_3 = 1 \qquad \therefore x_2 = 1.8125$$

$$x_1 - x_2 + 3x_3 = 2 \qquad \therefore x_1 = 1.1875$$

Note: The difficulty of Gaussian method is that sometimes you have to interchange rows and sometimes you will not have a unique answer to the

Gauss-Jordan Elimination:

A popular variant of Gaussian Elimination is Gauss-Jordan Elimination.

The idea is to reduce all elements in a column to zero except the diagonal element, Repeating this procedure to get

$$\left[\begin{array}{ccccc:c} a_{1.1} & 0 & 0 & \dots & 0 & b_1^{(n-1)} \\ 0 & a_{2.2} & 0 & \dots & 0 & b_2^{(n-1)} \\ \vdots & \vdots & a_{3.3} & \dots & 0 & b_3^{(n-1)} \\ 0 & 0 & 0 & \dots & a_{nn} & b_n^{(n-1)} \end{array}\right]$$

Example: Using the Gauss-Jordan elimination method solve

$$
\begin{aligned}
4x_1 + 2x_2 + 3x_3 &= 7 \\
2x_1 - 4x_2 - x_3 &= 1 \\
-x_1 + x_2 + 4x_3 &= -5
\end{aligned}
$$

Example:

$$
\tilde{A} = \left[\begin{array}{ccc:c} 4 & 2 & 3 & 7 \\ 2 & -4 & -1 & 1 \\ -1 & 1 & 4 & -5 \end{array}\right] \Rightarrow \left[\begin{array}{ccc:c} 4 & 2 & 3 & 7 \\ 0 & -5 & -\frac{5}{2} & \frac{-5}{2} \\ 0 & \frac{3}{2} & \frac{3}{2} & \frac{-13}{4} \end{array}\right]
$$

Now eliminate all elements in the second column except the diagonal element -5.

We want to eliminate 2 from the second column and eliminate $^3/_2$ from the third row to get

$$\begin{bmatrix} 4 & 0 & 2 & \vdots & 8 \\ 0 & -5 & 0 & \vdots & -\frac{5}{2} \\ 0 & 0 & 4 & \vdots & -4 \end{bmatrix}$$

Now, we want to eliminate 2 and $^{-5}/_2$ from the third column to get

$$\begin{bmatrix} 4 & 0 & 0 & \vdots & 8 \\ 0 & -5 & 0 & \vdots & -5 \\ 0 & 0 & 4 & \vdots & -4 \end{bmatrix}$$

Thus $x_3 = -1, \qquad x_2 = 1, \qquad x_1 = 2$

Note: The Gaussian Elimination method is more efficient.

Example:

Solve by Gauss-Jordan Elimination

$$
\begin{aligned}
x_1 + 3x_2 - 2x_3 + 2x_5 &= 0 \\
2x_1 + 6x_2 - 5x_3 - 2x_4 + 4x_5 - 3x_6 &= 1 \\
5x_3 + 10x_4 + 15x_6 &= 5 \\
2x_1 + 6x_2 + 8x_4 + 4x_5 + 18x_6 &= 6
\end{aligned}
$$

Solution:

The augmented matrix is

$$\tilde{A} = \left[\begin{array}{cccccc:c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 2 & 6 & -5 & -2 & 4 & -3 & -1 \\ 0 & 0 & 5 & 10 & 0 & 15 & 5 \\ 2 & 6 & 0 & 8 & 4 & 18 & 6 \end{array}\right] \Rightarrow$$

$$\left[\begin{array}{cccccc:c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & -1 & -2 & 0 & -3 & -1 \\ 0 & 0 & 5 & 10 & 0 & 15 & 5 \\ 0 & 0 & 4 & 8 & 0 & 18 & 6 \end{array}\right] \Rightarrow$$

$$\left[\begin{array}{cccccc:c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 6 & 2 \end{array}\right]$$

Interchanging the third and fourth rows to give

$$\left[\begin{array}{cccccc:c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 6 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Adding $^{-1}/_{2}$ times the third row to the second row and then adding 2 times the second row of the resulting matrix to the first row yields the reduced from

$$\begin{bmatrix} 1 & 3 & 0 & 4 & 2 & 0 & \vdots & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 & \vdots & 0 \\ 0 & 0 & 0 & 0 & 0 & 6 & \vdots & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & \vdots & 0 \end{bmatrix}$$

the corresponding system of equations is

$$x_1 + 3x_2 + 4x_4 + 2x_5 = 0$$

$$x_3 + 2x_4 = 0$$

$$6x_6 = 2 \qquad \therefore x_6 = 1/3$$

$$x_3 = 2x_4$$

$$x_1 = -3x_2 - 4x_4 - 2x_5$$

If we assume

x_2, x_4 and x_5 the arbitrary values γ, s and t respectively the solution is

$$x_1 = -3\gamma - 4s - 2t$$

$$x_2 = \gamma$$

$$x_3 = -2s$$

$$x_4 = s$$

$$x_5 = t$$

$$x_6 = \frac{1}{3}$$

H. W.:

Solve the given linear system A equations using either Gaussian Elimination or Gauss-Jordan Elimination:

(1)

$$\begin{aligned} -3x + 5y &= -22 \\ 3x + 4y &= 4 \\ 4x - 8y &= 32 \end{aligned}$$

(2)

$$\begin{aligned} x_1 + x_2 - 5x_3 &= 3 \\ x_1 \qquad - 5x_3 &= 1 \\ 2x_1 - x_2 - x_3 &= 0 \end{aligned}$$

(3)

$$\begin{aligned} x_1 - x_2 + 2x_3 + 2x_4 + 6x_5 &= 6 \\ 3x_1 - 2x_2 + 4x_3 + 4x_4 + 12x_5 &= 12 \\ x_2 - x_3 - x_4 - 3x_5 &= -3 \\ 2x_1 - 2x_2 + 4x_3 + 5x_4 + 15x_5 &= 10 \\ 2x_1 - 2x_2 + 4x_3 + 4x_4 + 13x_5 &= 13 \end{aligned}$$

CHAPTER – 2

Matrices

Definition:

A matrices can be denoted by a rectangular array of numbers

$$A = [a_{ij}] = \begin{bmatrix} a_{1.1} & a_{1.2} & \cdots & a_{1n} \\ a_{2.1} & a_{2.2} & \cdots & a_{2.n} \\ a_{3.1} & a_{3.2} & \cdots & a_{3n} \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

Definition:

If two matrices have the same size we can add them

If $A = \left[\alpha_{ij}\right], \quad B = \left[\beta_{ij}\right],$ then

$$(A + B)_{ij} = \alpha_{ij} + \beta_{ij}.$$

Example:

$$\begin{bmatrix} -1 & 2 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 3 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} -1+1 & 2+3 \\ 0-1 & 1+2 \end{bmatrix} = \begin{bmatrix} 0 & 5 \\ -1 & 3 \end{bmatrix}$$

Definition:

If A is any matrix and C is any scalar , then

Example:

If A is the matrix $A = \begin{bmatrix} 4 & 2 \\ 1 & 3 \\ -1 & 0 \end{bmatrix}$

then $2A = \begin{bmatrix} 8 & 4 \\ 2 & 6 \\ -2 & 0 \end{bmatrix}$ and $(-1)\ A = \begin{bmatrix} -4 & -2 \\ -1 & -3 \\ -1 & 0 \end{bmatrix}$

Definition:

If $A = [A_{ij}]$ is an m x n matrix and $B = [\beta_{ij}]$ is an n x p matrix then

The product AB is an m x p matrix

$$AB = \left[C_{ij}\right]$$

where

$$C_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj} - a_{i1} b_{ij} + a_{i2} b_{2j} + a_{i3} b_{3j} + a_{in} b_{nj}$$

Example:

Find the product AB where

$$A = \begin{bmatrix} -1 & 3 \\ 4 & -2 \\ 5 & 0 \end{bmatrix}$$

and

$$B = \begin{bmatrix} -3 & 2 \\ -4 & 1 \end{bmatrix}$$

Solution:

$$AB = \begin{bmatrix} -1 & 3 \\ 4 & -2 \\ 5 & 0 \end{bmatrix} \begin{bmatrix} -3 & 2 \\ -4 & 1 \end{bmatrix} = \begin{bmatrix} -ix\ -3+3x\ -4 & -1\times 2+3\times 1 \\ 4x\ -3-2x\ -4 & 4\times 2+-2x_{1} \\ 5x\ -3+ox\ -4 & 5\times 2+ox_{1} \end{bmatrix}$$

$$= \begin{bmatrix} -9 & 1 \\ -4 & 6 \\ -15 & 10 \end{bmatrix}$$

Definition:

The m x n matrix is square matrix containing I's down main diagonal, O's elements elsewhere is the identity matrix I_n

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Properties of the identity Matrix:

If A is a matrix of order m x n then the following properties are true:

1) $A \, I_n = A$
2) $I_m A = A$

The Transpose of a matrix:

The transpose of a matrix is formed by writing its columns as rows

$$e.g.\quad A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad then\ A^t = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The inverse of a matrix:

Definition:

$A \neq 0$

The square matrix A has an inverse iff det (A is a non singular) if $AB = BA = I_n$

$$then \quad B = A^{-1}$$

To Find inverse of a matrix A if it exists:

by adjoining the identity matrix to the coefficient matrix using row operation only (OR by column operation only).

$$e.g.\,(1)\; A = \begin{bmatrix} 0 & -3 & 2 & 1 \\ -1 & -2 & 1 & 1 \\ 5 & -3 & 4 & 0 \\ -1 & -3 & 2 & 3 \end{bmatrix}$$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 0 & -3 & 2 & 1 \\ 0 & 1 & 0 & 0 & -1 & -2 & 1 & 1 \\ 0 & 0 & 1 & 0 & 5 & -3 & 4 & 0 \\ 0 & 0 & 0 & 1 & -1 & -3 & 2 & 3 \end{array}\right]$$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 0 & -3 & 2 & 1 \\ 0 & 1 & 0 & 0 & -1 & -2 & 1 & 1 \\ 0 & 5 & 1 & 0 & 0 & -13 & 9 & 5 \\ 0 & -1 & 0 & 1 & 0 & -1 & 1 & 2 \end{array}\right]$$

$$\left[\begin{array}{cccc|cccc} 1 & 3 & 0 & -3 & 0 & 0 & -1 & -5 \\ 0 & 3 & 0 & -2 & -1 & 0 & -1 & -3 \\ 0 & 18 & 1 & 13 & 0 & 0 & -4 & -21 \\ 0 & -1 & 0 & 1 & 0 & -1 & 1 & 2 \end{array}\right]$$

$$\left[\begin{array}{cccc|cccc} 1 & 3 & 0 & -3 & 0 & 0 & -1 & -5 \\ -1 & 0 & 0 & 1 & -1 & 0 & 0 & 2 \\ -4 & 6 & 1 & -1 & 0 & 0 & 0 & -1 \\ 1 & 2 & 0 & -2 & 0 & -1 & 0 & -3 \end{array}\right]$$

$$\left[\begin{array}{cccc|cccc} 21 & -27 & -5 & 2 & 0 & 0 & -1 & 0 \\ -9 & 12 & 2 & -1 & -1 & 0 & 0 & 0 \\ -4 & 6 & 1 & -1 & 0 & 0 & 0 & -1 \\ 13 & -16 & -3 & 1 & 0 & -1 & 0 & 0 \end{array}\right]$$

$$\left[\begin{array}{cccc|cccc} 9 & -12 & -2 & 1 & 1 & 0 & 0 & 0 \\ -13 & 16 & 3 & -1 & 0 & 1 & 0 & 0 \\ -21 & 27 & 5 & -2 & 0 & 0 & 1 & 0 \\ 4 & -6 & -1 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\therefore A^{-1} = \begin{bmatrix} 9 & -12 & -2 & 1 \\ -13 & 16 & 3 & -1 \\ -21 & 27 & 5 & -2 \\ 4 & -6 & -1 & 1 \end{bmatrix}$$

(2) $$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 1 & -1 \\ 0 & 1 & 0 & 5 & 4 & 1 \\ 0 & 0 & 1 & 4 & 2 & 2 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 1 & -1 \\ -5 & 1 & 0 & 0 & -1 & 6 \\ -4 & 0 & 1 & 0 & -2 & 6 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 1 & -1 \\ -5 & 1 & 0 & 0 & -1 & 6 \\ \frac{-4}{6} & 0 & \frac{1}{6} & 0 & \frac{-2}{6} & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} \frac{2}{6} & 0 & \frac{1}{6} & 1 & \frac{4}{6} & 0 \\ -1 & 1 & -1 & 0 & 1 & 0 \\ \frac{-4}{6} & 0 & \frac{-1}{6} & 0 & \frac{-2}{6} & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} \frac{2}{6} & 0 & \frac{1}{6} & 1 & \frac{4}{6} & 0 \\ -1 & 1 & -1 & 0 & 1 & 0 \\ \frac{-10}{6} & 1 & \frac{-5}{6} & 0 & \frac{4}{6} & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} \frac{2}{6} & 0 & \frac{1}{6} & 1 & \frac{4}{6} & 0 \\ -1 & 1 & -1 & 0 & 1 & 0 \\ \frac{-6}{6} & \frac{2}{6} & \frac{-1}{6} & 0 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} \frac{6}{6} & \frac{-4}{6} & \frac{5}{6} & 1 & 0 & 0 \\ -1 & 1 & -1 & 0 & 1 & 0 \\ \frac{-6}{6} & \frac{2}{6} & \frac{-1}{6} & 0 & 0 & 1 \end{array}\right]$$

$$\therefore A^{-1} = \begin{bmatrix} 1 & \frac{-4}{6} & \frac{5}{6} \\ -1 & 1 & -1 \\ -1 & \frac{2}{6} & \frac{-1}{6} \end{bmatrix}$$

Calculation of inverses of square matrix:

We give a way of calculating A^{-1} if it exists.

Definition:

A djoint matrix suppose $A = (\alpha_{ij})$ we define

the a djoint of $A, A^* = (\alpha_{ij}^*)$ when α_{ij} is the

$$\alpha_{ij}^* = (1)^{j+i}\alpha_{ji}$$

cofactor of α_{ij} $\Lambda A.$

Thus A* is the transpose of the matrix it co-factor

$$e.g.: \quad A = \begin{bmatrix} 1 & 2 & 1 \\ 4 & 2 & 0 \\ 1 & 2 & 3 \end{bmatrix}$$

$$A^* = \begin{bmatrix} \begin{vmatrix} 1 & 0 \\ 2 & 3 \end{vmatrix} & -\begin{vmatrix} 2 & 1 \\ 2 & 3 \end{vmatrix} & \begin{vmatrix} 2 & 1 \\ 1 & 0 \end{vmatrix} \\ -\begin{vmatrix} 4 & 0 \\ 1 & 3 \end{vmatrix} & -\begin{vmatrix} 1 & 1 \\ 1 & 3 \end{vmatrix} & -\begin{vmatrix} 1 & 1 \\ 4 & 0 \end{vmatrix} \\ \begin{vmatrix} 4 & 1 \\ 1 & 2 \end{vmatrix} & -\begin{vmatrix} 1 & 2 \\ 1 & 2 \end{vmatrix} & \begin{vmatrix} 1 & 2 \\ 4 & 1 \end{vmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} 3 & -4 & -1 \\ -12 & 2 & 0 \\ 7 & 0 & -7 \end{bmatrix}$$

$$\textit{Consider} \qquad AA^* = \begin{bmatrix} -14 & 0 & 0 \\ 0 & -14 & 0 \\ 0 & 0 & -14 \end{bmatrix}$$

$$A^*A = \begin{bmatrix} -14 & 0 & 0 \\ 0 & -14 & 0 \\ 0 & 0 & -14 \end{bmatrix}$$

$$\therefore \; AA^* = A^*A = -14 I_3$$

$$\therefore -14 = \det A$$

$$\text{if} \quad B = -\frac{1}{14} \quad A^* = \begin{bmatrix} \frac{-3}{14} & \frac{+4}{14} & \frac{1}{14} \\ \frac{+12}{14} & \frac{-2}{14} & \frac{-11}{14} \\ \frac{-7}{14} & 0 & \frac{7}{14} \end{bmatrix}$$

$$\therefore AB = BA = I_3 \qquad \therefore B = A^{-1}$$

Theorem:

For any n x n matrix A we hare

$$AA^* = A^*A = \det A \, . I_n$$

Corally:

If A is any n x n matrix with $\det A \neq 0$

then A is non-singular and

$$A^{-1} = \frac{1}{\det A} A^*$$

Example:

$$A = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \quad \det A = 1\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} + 1\begin{vmatrix} -1 & 1 \\ 1 & 1 \end{vmatrix} = -2$$

A^{-1} *esists*.

$$\therefore A^* = \begin{bmatrix} \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} & -\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} \\ -\begin{vmatrix} -1 & 1 \\ 1 & 1 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} & -\begin{vmatrix} 1 & 1 \\ -1 & 1 \end{vmatrix} \\ \begin{vmatrix} -1 & 1 \\ 1 & 1 \end{vmatrix} & -\begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} & \begin{vmatrix} 1 & 0 \\ -1 & 1 \end{vmatrix} \end{bmatrix} = \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix}$$

$$\therefore A^{-1} = \frac{-1}{2} \quad A^* = \begin{vmatrix} 0 & \frac{-1}{2} & \frac{-1}{2} \\ -1 & 0 & 1 \\ 1 & \frac{1}{2} & \frac{-1}{2} \end{vmatrix}$$

$$\begin{pmatrix} \det A \neq 0 & \Rightarrow & A^{-1}\ exists \\ A^{-1}\ exists & \Rightarrow & \det A \neq 0 \end{pmatrix}$$

Properties of inverse of matrices:

(1) $\left(A^{-1}\right)^{-1} = A$

(2) $(cA)^{-1} = \frac{1}{2} A^{-1}$ $\quad$ *where c is scalax .*

(3) $\left(A^{t}\right)^{-1} = \left(A^{-1}\right)^{t}$

The inverse Product:

Theorem: If A and B are invertible of under n the AB is invertible and

$$(AB)^{-1} = B^{-1} A^{-1}$$

System of equations:

Theorem:

If A is an invertible matrix, then the system of linear equations represented by

Ax = B has a unique solution given by

$$x = A^{-1}B. \qquad \begin{pmatrix} An = B \\ A^{-1}Ax = A^{-1}B \\ Ix = A^{-1}B \end{pmatrix}$$

Example:

Solve the system of equations wing an

$$2x + 3y + Z = -1$$
$$3x + 3y + Z = 1$$
$$2x + 4y + Z = -2$$

Solution:

$$A = \begin{bmatrix} 2 & 3 & 1 \\ 3 & 3 & 1 \\ 2 & 4 & 1 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 6 & -2 & -3 \end{bmatrix}$$

To solve x,

$$x = A^{-1}B = \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 6 & -2 & -3 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \\ -2 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ Z \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ -2 \end{bmatrix}$$

$$\therefore x = 2, \quad y = -1, \quad Z = -2$$

Elementary matrices:

Elementary matrices are of three types:

(i) P_{ij} is determined by interchanging i^{th} row and j^{th} row.

$$P_{ij} = \begin{bmatrix} 1 & & & & & \\ & 0 & \cdots & 1 & & \\ & & 1 & & & \\ & & & 1 & & \\ & & & & 0 & \\ & & & & & 1 \end{bmatrix} \begin{matrix} \\ \leftarrow i \\ \\ \\ \to j \\ \\ \end{matrix}$$

$M_i(\lambda)$ i^{th} $\lambda \neq 0$

$I_n)$

(ii) is obtained by multiplying row by

$$M_i(\lambda) = \begin{bmatrix} \lambda & \cdots & \bigcirc \\ & \lambda & \\ \bigcirc & & \lambda \end{bmatrix} \qquad \det M_i(\lambda) = \lambda$$

(iii) $\delta_{ij}(\lambda)$ is obtained from I by adding λ times j^{th} row to i^{th} row.

$$\delta_{ij}(\lambda) = \begin{bmatrix} 1 & & & & \\ & 1 & & \bigcirc & \\ & & 1 & \lambda \cdots & \\ & \bigcirc & & 1 & \\ & & & & 1 \end{bmatrix} \begin{matrix} \\ \\ \leftarrow i^{th} \; row \\ \leftarrow j^{th} \; row \\ \\ \end{matrix}$$

$\delta_{ij}(\lambda)$ λ $(i, j)^{th}$

Theorem:

If E is an elementary matrix OR square diagonal matrix and it A a square matrix of the same size then

$$\det(EA) = \det E \ \det A.$$

$$\det(AE) = \det A \det E.$$

Theorem:

Let A be m x n matrix then $\exists$ a product p of m x m elementary matrices and a product Q of n x n elementary matrices such that PAQ is a

Example:

Suppose $A = \begin{bmatrix} 1 & -2 & 5 & 1 \\ 1 & 1 & -7 & -2 \\ -1 & 8 & -29 & -7 \end{bmatrix}$

then to find P and Q.

$$\underset{Row\ \ operations}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \begin{bmatrix} 1 & -2 & 5 & 1 \\ 1 & 1 & -7 & -2 \\ -1 & 8 & -29 & -7 \end{bmatrix} \underset{Column\ \ Operations}{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -2 & 5 & 1 \\ 1 & 3 & -12 & -3 \\ 0 & 6 & -24 & -6 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 3 & -12 & -3 \\ 0 & 6 & -24 & -6 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 3 & -12 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & -5 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & -5 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & -6 & 25 & 6 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & -5 & -1 \\ 0 & 1 & 12 & 3 \\ 0 & -6 & 25 & 6 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\therefore \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}}_{P} \underbrace{\begin{bmatrix} 1 & -2 & 5 & 1 \\ 1 & 1 & -7 & -2 \\ -1 & 8 & -29 & -7 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} 1 & 2 & -5 & -1 \\ 0 & 1 & 12 & 3 \\ 0 & -6 & 25 & 6 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{Q} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Theorem:

If a square matrices F, G, H, … etc have inverses and are of the same size then their product FGH … etc has an inverse and $(FGHJ\ldots)^{-1} = \ldots J^{-1}H^{-1}G^{-1}F^{-1}$.

Theorem:

Every elementary matrix has an inverse

Exercise:

(1) Show that B is the inverse of A

$$A = \begin{bmatrix} -2 & 2 & 3 \\ 1 & -1 & 0 \\ 0 & 1 & 4 \end{bmatrix}, \quad B = \frac{1}{3}\begin{bmatrix} -4 & -5 & 3 \\ -4 & -8 & 3 \\ 1 & 2 & 0 \end{bmatrix}$$

(2) Find the inverse of the matrix (if it exists)

$$(a) \quad \begin{bmatrix} 1 & 2 & -1 \\ 3 & 7 & 10 \\ 7 & 16 & 21 \end{bmatrix}, \quad b \begin{bmatrix} 1 & 2 & 2 \\ 3 & 7 & 9 \\ -1 & -4 & -7 \end{bmatrix}$$

(3) Use an inverse matrix to solve the given system of linear equations

$$\begin{aligned} x_1 - 2x_2 - x_3 - 2x_4 &= 0 \\ 3x_1 - 5x_2 - 2x_3 - 3x_4 &= 1 \\ 2x_1 - 5x_2 - 2x_3 - 5x_4 &= -1 \\ -x_1 + 4x_2 + 4x_3 + 11x_4 &= 2 \end{aligned}$$

(4) For each of the following matrices A and B find the product of elementary matrices (P,Q,R, and S) such that PAQ and RBS are diagonal matrices.

$$A = \begin{bmatrix} -1 & 2 & 7 & 3 \\ 3 & 0 & 9 & 3 \\ 5 & 2 & 5 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 3 & 6 & 3 \\ 2 & 3 & 1 \\ -2 & -2 & -6 \\ 0 & 1 & -5 \end{bmatrix}$$

CHAPTER THREE

Determinants:

The equations:

$$a_{1.1}x_1 + a_{1.2}x_2 = b_1$$
$$a_{2.1}s_2 + a_{2.2}s_2 = b_2$$

have a unique solution iff

$$a_{1.1}a_{2.2} - a_{1.2}a_{2.1} \neq 0$$

this quantity is called the determinant of the coefficient matrix A.

$$A = \begin{bmatrix} a_{1.1} & a_{1.2} \\ a_{2.1} & a_{2.2} \end{bmatrix}$$

Definition of the determinant of a 2 x 2 matrix:

is given by

$$\det A = \begin{vmatrix} a_{1.2} & a_{1.2} \\ a_{2.1} & a_{2.2} \end{vmatrix} = a_{1.1}a_{2.2} - a_{1.2}a_{2.1}$$

Definition:

If A is a square of order 2 OR grater, then the determinant of A is the sum of the entries in the first row of A multiplied by their cofactors.

Example:

$$A = \begin{vmatrix} 0 & 2 & 1 \\ 3 & -1 & 2 \\ 4 & 0 & 1 \end{vmatrix}$$

then

$$\det A = \begin{vmatrix} 0 & 2 & 1 \\ 3 & -1 & 2 \\ 4 & 0 & 1 \end{vmatrix} = 0\begin{vmatrix} -1 & 2 \\ 0 & 1 \end{vmatrix} - 2\begin{vmatrix} 3 & 2 \\ 4 & 1 \end{vmatrix} + \begin{vmatrix} 3 & -2 \\ 4 & 0 \end{vmatrix}$$

$$= 0 - 2\left(3-8\right) + 1(+4)$$

$$= 10 + 4 = 14$$

Example:

$$A = \begin{vmatrix} 1 & -2 & 3 & 0 \\ -1 & 1 & 0 & 2 \\ 0 & 2 & 0 & 3 \\ 3 & 4 & 0 & -2 \end{vmatrix}$$

$$then \;\; \det A = \begin{vmatrix} 1 & -2 & 3 & 0 \\ -1 & 1 & 0 & 2 \\ 0 & 2 & 0 & 3 \\ 3 & 4 & 0 & -2 \end{vmatrix} = 1\begin{vmatrix} 1 & 0 & 2 \\ 2 & 0 & 3 \\ 4 & 0 & -2 \end{vmatrix}$$

$$+2\begin{vmatrix} -1 & 0 & 2 \\ 0 & 0 & 3 \\ 3 & 0 & -2 \end{vmatrix} + 3\begin{vmatrix} -1 & 1 & 2 \\ 0 & 2 & 3 \\ 3 & 4 & -2 \end{vmatrix}$$

$$= 1\left\{1\begin{vmatrix} 0 & 3 \\ 0 & -2 \end{vmatrix} - 0\begin{vmatrix} 2 & 3 \\ 4 & -2 \end{vmatrix}\right\} + 2\left\{-1\begin{vmatrix} 0 & 3 \\ 0 & -2 \end{vmatrix} - 0\begin{vmatrix} & \\ & \end{vmatrix} + 2\begin{vmatrix} 0 & 0 \\ 3 & 0 \end{vmatrix}\right\}$$

$$+3\left\{-1\begin{vmatrix}2 & 3\\ 4 & -2\end{vmatrix}-1\begin{vmatrix}0 & 3\\ 3 & -2\end{vmatrix}\right\}+2\left\{\begin{vmatrix}0 & 2\\ 3 & 4\end{vmatrix}\right\}$$

$$=3\{-1(-4-12)-(-9)+2(-6)\}=3(16+9-12)=39$$

The determinant of a triangular Matrix:

Example:

$$A=\begin{bmatrix}2 & 0 & 0 & 0\\ 4 & -2 & 0 & 0\\ -5 & 6 & 1 & 0\\ 1 & 5 & 3 & 3\end{bmatrix}$$

$$\det A = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 4 & -2 & 0 & 0 \\ -5 & 6 & 1 & 0 \\ 1 & 5 & 3 & 3 \end{bmatrix} = 2\begin{bmatrix} 2 & 0 & 0 \\ 6 & 1 & 0 \\ 5 & 3 & 3 \end{bmatrix}$$

$$= 2\left\{-2\begin{vmatrix} 1 & 0 \\ 3 & 3 \end{vmatrix}\right\} = 2\left(-2\right)(3)(1)$$

$$= -12$$

Example:

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & -2 \end{bmatrix}$$

$$\det A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2 \end{bmatrix} = 1 \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix}$$

$$= 1 \left\{ 3 \begin{vmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & -2 \end{vmatrix} \right\} = (1)(3)(2) \begin{vmatrix} 4 & 0 \\ 0 & -2 \end{vmatrix}$$

$$= (1)(3)(2)(4)(-2) = -48$$

Evaluation of a determinant using elementary operations:

By elementary row operations we not that

1. Interchanging two rows of the matrix changed the sign of its determinant.
2. Multiplying a row by a non zero constant multiplied the determinant by that same constant.

Theorem:

Condition that yield a zero determinant

If A is a square matrix and any of the following

(1) An entire row (OR column) consists of zeros.

(2) Two rows (OR columns) are equal.

(3) One row (OR column) is a multiple of another row (OR column).

Properties of determinants:

1. If A and B are square matrices of order n, then $|AB|=|A|\,|B|$.

2. If A is an n x n matrix and C is scalar, then $|cA|=C^n|A|$.

3. $|A|+|B| \neq |A+B|$

4. If A is invertible then

$$|A^{-1}| = \frac{1}{|A|}$$

5. If A is a square matrix, then

$$\left|A\right| = \left|A^{t}.\right|$$

Applications of Determinants:

1) Cramers Rule:

Is a formula that uses determinants to solve a system of n linear equations in n variables.

This rule can be applied only to systems of consider two linear equations in two unknowns.

$$a_{1.1}x_1 + a_{1.2}x^{2} = b_1$$

$$a_{2.1}x_1 + a_{2.2}x_2 = b_2$$

$$then \quad x_1 = \frac{a_{2.2}b_1 - a_{1.2}b_2}{a_{1.1}a_{2.2} - a_{2.1}a_{1.2}}$$

recognizing that the numerator and denominator

for both x_1 and x_2 can be represented as

determinants we have

$$x_1 = \frac{\begin{vmatrix} b_1 & a_{1.2} \\ b_2 & a_{2.2} \end{vmatrix}}{\begin{vmatrix} a_{1.1} & a_{1.2} \\ a_{2.1} & a_{2.2} \end{vmatrix}} \quad , \quad x_2 = \frac{\begin{vmatrix} a_{1.1} & b_1 \\ a_{2.1} & b_2 \end{vmatrix}}{\begin{vmatrix} a_{1.1} & a_{1.2} \\ a_{2.1} & a_{2.2} \end{vmatrix}}$$

provided

$$a_{1.1}a_{2.2} - a_{2.1}a_{1.2} \neq 0$$

Denoting:

$$\Delta \quad = \begin{vmatrix} a_{1.1} & a_{1.2} \\ a_{2.1} & a_{2.2} \end{vmatrix}$$

$$\Delta x_1 = \begin{vmatrix} b_1 & a_{1.2} \\ b_2 & a_{2.2} \end{vmatrix}$$

$$\Delta x_2 = \begin{vmatrix} a_{1.1} & b_1 \\ b_{2.1} & b_2 \end{vmatrix}$$

$$x_1 = \frac{\Delta x_1}{\Delta}, \qquad x_2 = \frac{\Delta x_2}{\Delta}$$

for n linear equations,

e.g./

$$a_{1.1}x_1 + a_{1.2}x_2 + a_{1.3}x_3 = b_1$$

$$a_{2.1}x_1 + x_{2.2}x_2 + a_{2.3}x_3 = b_2$$

$$a_{3.1}x_1 + a_{3.2}x_2 + a_{3.3}x_3 = b_3$$

$$\Delta \begin{vmatrix} a_{1.1} & a_{1.2} & a_{1.3} \\ a_{2.1} & a_{2.2} & a_{2.3} \\ a_{3.1} & a_{3.2} & a_{3.3} \end{vmatrix} \qquad x_3 = \frac{\Delta x_3}{\Delta}$$

$$\Delta x_3 = \begin{vmatrix} a_{1.1} & a_{1.2} & b_1 \\ a_{2.1} & a_{2.2} & b_2 \\ a_{3.1} & a_{3.2} & b_3 \end{vmatrix}$$

Area of Triangle in the x y - Plane:

The area of the triangle whose vertices are (x1, y1), (x_2, y_2) and (x_3, y_3) is given by

Area = $$\pm\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$$

when the sign (±) is chosen to give a positive area

Example: Find the area of the triangle whose vertices are (1,0), (2, 2) and (4, 3)

Solution:

Area = $\pm\frac{1}{2}\begin{vmatrix} 1 & 0 & 1 \\ 2 & 2 & 1 \\ 4 & 3 & 1 \end{vmatrix} = \frac{3}{2}$

Test for collinear points in the xy - Plane:

Three points (x1, y1), (x2, y2) and (x3, y3) are collinear iff:

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$$

3.Equation of a line passing through two points:

e. g Finding the equation of the line passing through the points (2,4) and (-1, 3) is given by:

$$\begin{vmatrix} x & y & 1 \\ 2 & 4 & 1 \\ -1 & 3 & 1 \end{vmatrix} = 0 \quad i.e. \quad x\begin{vmatrix} 4 & 1 \\ 3 & 1 \end{vmatrix} - y\begin{vmatrix} 2 & 1 \\ -1 & 1 \end{vmatrix} + 1\begin{vmatrix} 2 & 4 \\ -1 & 3 \end{vmatrix} = 0$$

x-3y = -10

4.Volume of Tetrahedron

The volume tetrahedron whose vertices are (x1, y1, z1), (x2, y2, z2), (x3, y3, z3) is give by :

$$\text{Volume} = \pm\frac{1}{6}\begin{vmatrix} x_1, & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix}$$

Where the sing (±) is chosen to give a positive volume.

Test for coplanar points in space:

Four points (x 1, y1, z1), (x 2, y2, z2), (x 3, y3, z3) and (x 4, y4, z4) are coplanar iff:

$$\begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix} = 0$$

5.Equation of a plane passing through three points:

The equation of the plane passing through the pints (x 1, y1, z1), (x 2, y2, z2), (x 3, y3, z3) is give by:

$$\begin{vmatrix} x & y & z & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{vmatrix} = 0$$

Exc.:

1) Show that

$$\begin{bmatrix} a & 1 & 1 & 1 \\ 1 & a & 1 & 1 \\ 1 & 1 & 1 & a \end{bmatrix} = (a+3)(a-1)^3$$

***2) Use cramers rule to find the solution of the system of linear* equation**

$$3 = 2x_1 + 3x_2 + 3x_3$$

$$13 = 6x_1 + 6x_2 + 12x_3$$

$$2 = 12x_1 + 9x_2 - x_3$$

3) Find the determinant of the following n x n matrix

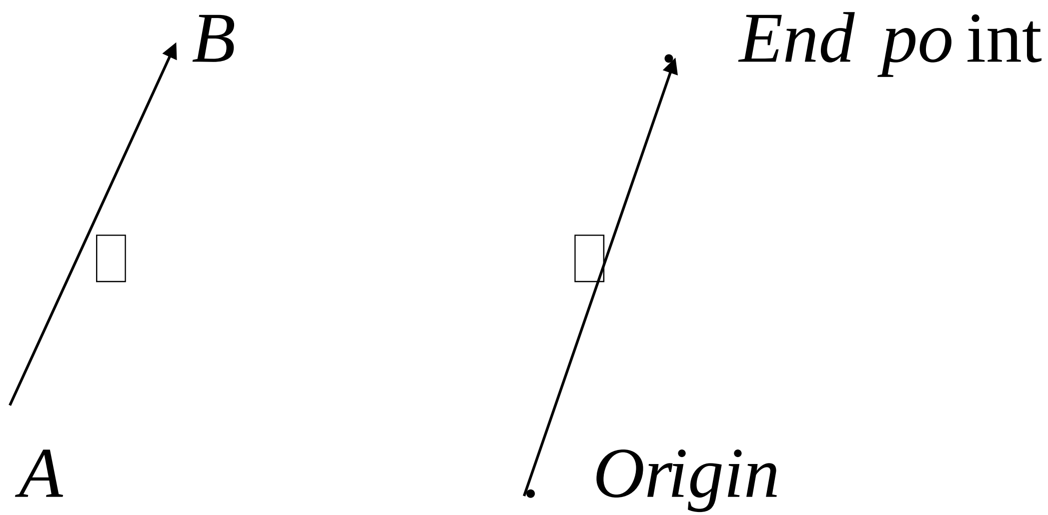
$$\begin{bmatrix} 1-n & 1 & 1 & \dots & 1 \\ 1 & 1-n & 1 & \dots & 1 \\ 1 & 1 & 1-n & \dots & 1 \\ 1 & 1 & 1 & \dots & 1-n \end{bmatrix} = 0$$

CHAPTER – 4

Vector Space

Vectors: 4.1)

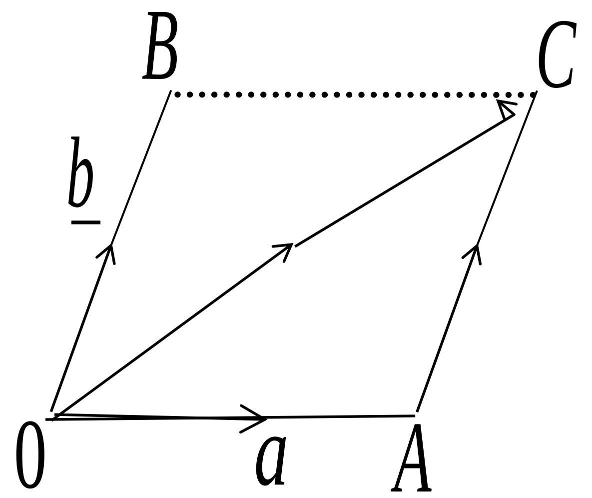
The notation of a vector $\overrightarrow{AB}$ or a



Zero vector has zero length , its direction is anywhere we like.
Two basic operations that can be performed with vectors

(i) Addition :

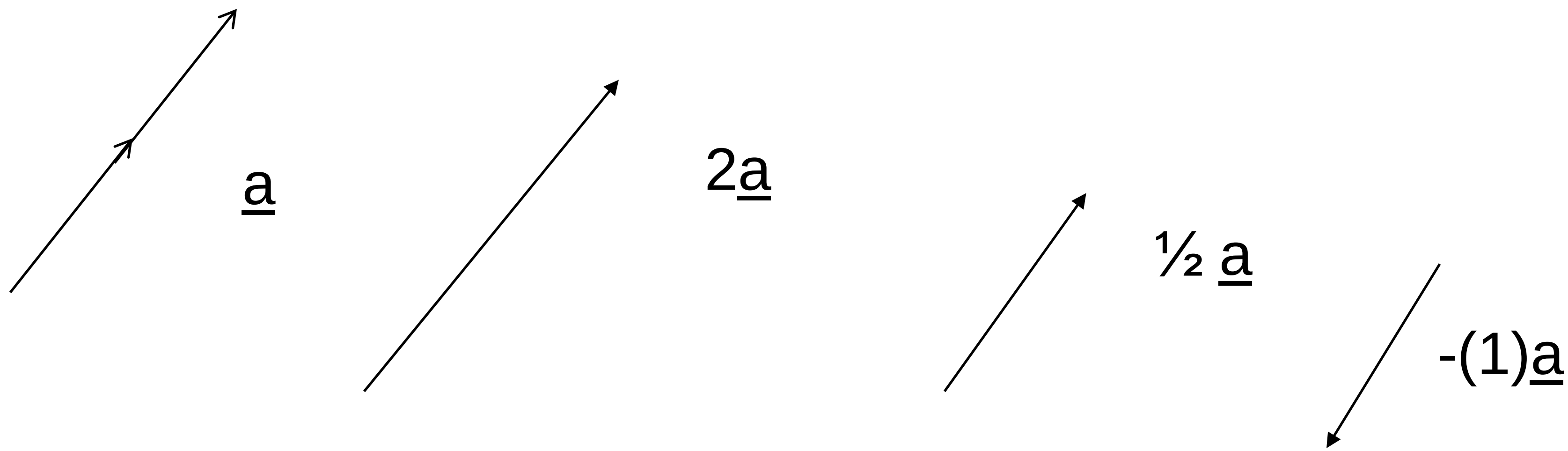
$$\underline{b} = \overrightarrow{oc} \text{ (Triangle-rule)}$$



(parallegram rule)

(ii) Multiplication by real number

Ka



Basic properties:

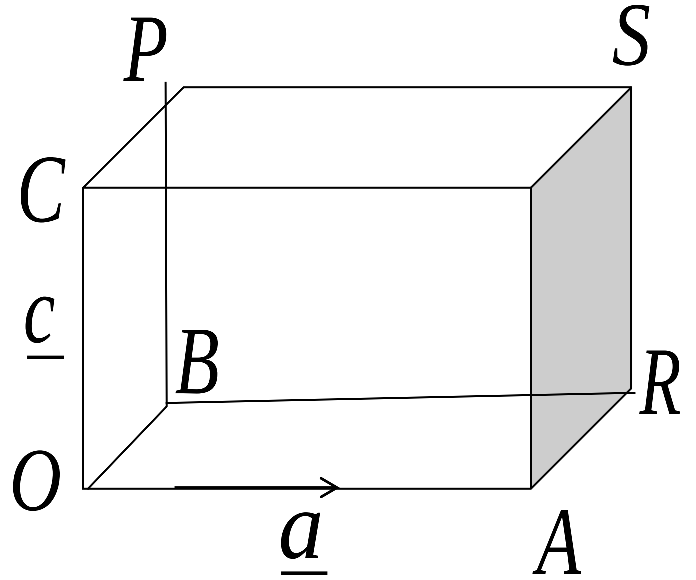
A1: $\underline{a} + \underline{b} = \underline{b} + \underline{a}$ (obvious from paralle gram rule)

A2: $(\underline{a} + \underline{b}) + \underline{C} = \underline{a} + (\underline{b} + \underline{c})$

$$(\underline{a} + \underline{b}) + \underline{C} = \overline{OR} + \overline{OR} = \overline{OS}$$

$$\underline{a} + (\underline{b} + \underline{c}) = \overrightarrow{OA} + \overrightarrow{OP}$$

$$= \overrightarrow{OA} + \overrightarrow{AS} = \overrightarrow{OS}$$

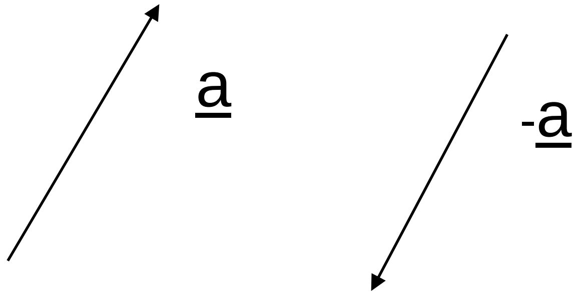


A3: $\underline{a} + \underline{0} = \underline{a}$

A4 : Given any vector $\underline{a}$ -∃ a unique a* such that $\underline{a} + \underline{a}^* = \underline{O}$

We normally denote $\underline{a}$ by $-\underline{a}$

$\underline{a} + (\underline{a}^*) = \underline{0}$



Scalar multiplication properties:

S1 : $(\propto + \beta)\underline{x} = \propto \underline{x} + \beta \underline{x}$

S2 : $\propto (\underline{x} + \underline{y}) = \propto \underline{x} + \propto \underline{y}$

S3 : $\alpha(\beta \underline{x}) = (\alpha\beta)\underline{x}$

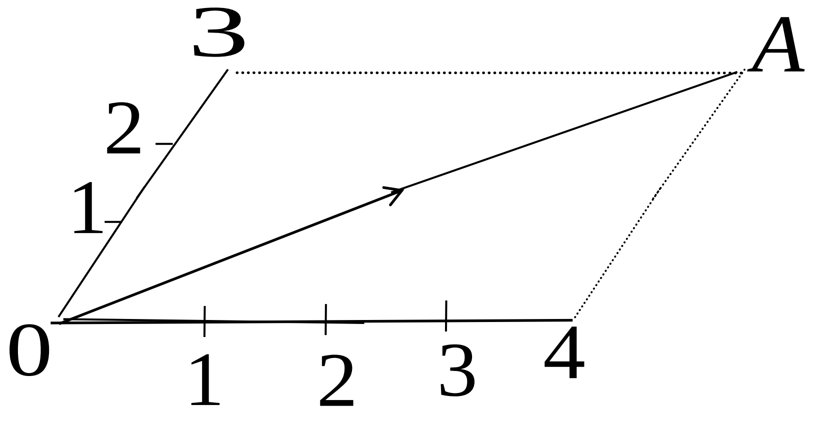
S4 : $\propto(\beta \underline{x}) = (\propto \beta)\underline{x}$

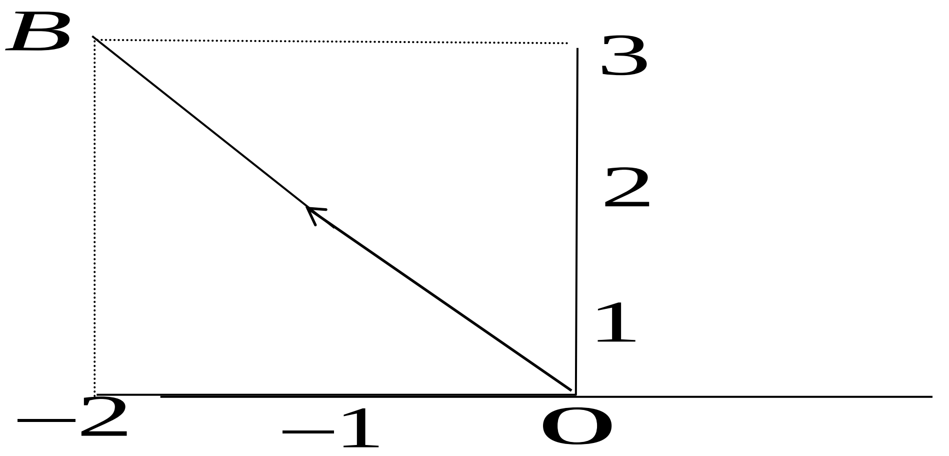
Vector in Plane:

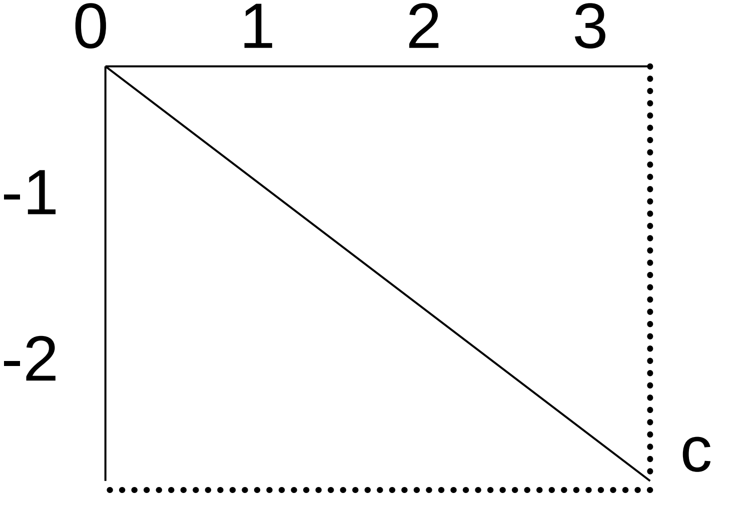
Choose an origin and axes (not necessarily at right angle) and a unit length, then every vector is represented by a pair of coordinates.

$$\overrightarrow{OA} = (4,3)$$

$$\overrightarrow{OB} = (-2,3)$$







$\overrightarrow{OC} = (3,-2)$ $\rightarrow$

$\overrightarrow{OX} = (3, 1)$

$\overrightarrow{Oy} = (1, 2)$

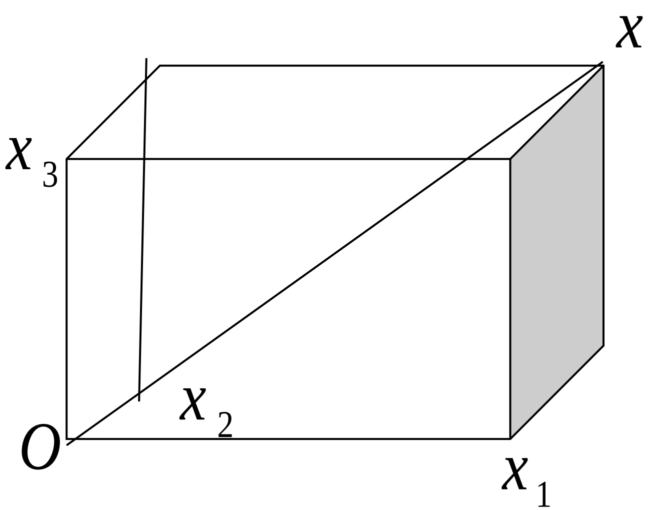
$\therefore \overrightarrow{OX} + \overrightarrow{OY} = (3,1) + (1, 2)$

$= (4, 3)$

If $Ox_1 = x_1$

$Ox_2 = x_2$

$Ox_3 = x_3$



Than $\overrightarrow{Ox}$ has coordinates x1, x2, x3

With respect to the axes through 0

$\therefore \quad \overrightarrow{Ox} = (x_1, x_2, x_3)$

If $\underline{x} = (x1, x2, x3)$

$\underline{y} = (y1, y2, y3)$

Then $\underline{x} + \underline{y} = (x1 + y1, x2 + y2, x3 + y3)$

And $K\underline{x} = (Kx1, Kx2, Kx3)$

Note : $\underline{O} = (0, 0, 0)$

Example : Find the sum of the following vectors $\underline{u} = (1, 4)$

$\underline{v} = (2, -2)$

Solution:

$\underline{u} + \underline{v} \quad = (1, 4) + (2, -2) \quad (3, 2)$

We can use * to prove A1- A4 and S1 – S4 e. g/:

SI : $$(\propto + \beta)\underline{x} = \propto \underline{x} + \beta \underline{x}$$

Then $$(\propto + \beta)\underline{x} = \{(\propto + \beta)x_1, (\propto + \beta)x_2, (\propto + \beta)x_3\}$$

$$= (\propto x_1, \propto x_2, \propto x_3) + (\beta x_1, \beta x_2, \beta x_3)$$

$$= \propto (x_1, x_2, x_3) + \beta(x_1, x_2, x_3)$$

$$= \propto \underline{x} + \beta \underline{x}$$

Using formula * and basic properties A1-A4 and S1-S4. We prove all the algebraic properties of vectors.

So, instead of saying (x1, x2, x3) represent a rector <u>or</u> are the coordinates vector we can say (x1, x2, x3).

Example : Let $\underline{U}$ = (2, -1, 5, 0), $\underline{V}$ = (4, 3 ,1, -1) and $\underline{W}$ = (-6, 2, 0, 3) solve for X as

$\underline{x} = 2\underline{u} – (\underline{v} + 3\underline{w})$

Solution :

$\underline{x}$ = 2 $\underline{u}$ = ($\underline{v}$+ 3$\underline{w}$)
= (4, -2, 10, 0) – (4, 3, 1,- 1) – (- 18, 6, 0, 9)
= (4 - 4 + 18, - 2 -3 - 6, 10 -1 -0, 0 + 1 - 9)
= (18, - 11, 9 - 8)

1.1) If $\underline{x} + \underline{y} = \underline{x} + \underline{z}$ then $\underline{y} = \underline{z}$

Proof.: suppose $\underline{x} + \underline{y} = \underline{x} + \underline{z}$

The vector – $\underline{x}$ exists (A4)

Then $(-\underline{x}) + (\underline{x} + \underline{y}) = (-\underline{x}) + (\underline{x} + \underline{z})$

$(-\underline{x} + \underline{x}) + \underline{y} = (-\underline{x} + \underline{x}) + \underline{z}$ (A2)

$= \underline{O} + \underline{z} \quad \underline{O} + \underline{y}$ (A1, A4)

$\underline{y} = \underline{z}$ (A1, A3)

1.2) $O\underline{x} = \underline{O}$

Proof. :

$\underline{x} + \underline{O} = 1\underline{x} = (1+0)\,\underline{x}$

$= 1\underline{x} + O\underline{x}$

$= \underline{x} + O\underline{x}$

$\underline{O} = O\underline{x}$

1.3. $$(\propto \underline{x}) = -(\propto)\underline{x}$$

In particular $-\underline{x} = (-1)\,\underline{x}$

1.4. $$(\propto - \beta)\underline{x} = \propto \underline{x} - (\beta \underline{x})$$

n – Vectors:

Definition: An ordered set (x1, x2, …. xn) of then real numbers is called an n-vector, we cannot give a geometrical interpolation of n-vectors in physical space when n > 3.

The sum of $\underline{x}$ = (x1, x2 ……… xn) and

$\underline{y}$ = (y1, y2 ………. yn)

Is defined to be

(x1 + y1, x2, + y2……. xn + yn)

And is defined by $\underline{x} + \underline{y}$

The product of scales α and x is

$$\propto \underline{x} = (\propto x_1, \propto x_2 \ldots\ldots\ldots \propto x_n)$$

The set of all n-vector is denoted by Rn

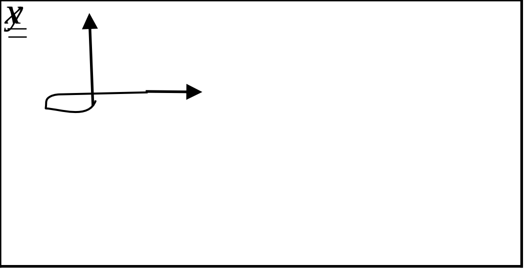
A1 – A4, S1 – S4 are true for n-vectors.

4.2) Sub spaces: –

Rn is called a vector space R^3 consist of all vectors in 3-space with a common origin $\underline{O}$ consider a subset S of R3 consisting of all vector lying in a plane through O.

We call S a sub space of R^3 (We regard a plane at a 2-dimensional Space)

What algebraic properties does S have?



(i) $\underline{x}, \underline{y} \in S \Rightarrow \underline{x} + \underline{y} \in S$

(II) $\underline{x} \in S \Rightarrow \alpha\, \underline{x} \in S$ where α is ascalar

Are there other types of sub set of R^3 that satisfy (i) and (ii)? Yes. The set of all vectors lying in a line through O.

Definition

Any subset set S of R^3 satisfying (i) and (ii) is either R^3 itself.

all vector in a plane through O OR

all vector in a line through O OR

O alone .

Example:

(i) The set of all 3-vectors type (x_1, x_2, O) is a subset of R^3.

(ii) The set of all 4-vectors type (x_1, x_1, x_2, x_3) is a subset of R4.

(iii) The set of all vectors type (x_1, x_2) is not a subspace of R^3.

Since (1, 1) $\in$ set

(2, 4) set

but (1, 1) + (2, 4) = (3, 5) $\notin$ set.

Example:

Which of the following subset is a subspace of R^3.

(i) $\underline{W} = (x1, x2, 1)$ X1 , x2 } rechnumber

(ii) $\underline{W} = (x1, x1 + x3, x3)$ X1 , x3

Solution :

(i) Since $\underline{O} = (O, O, O)$ is not in $\underline{W}$ then $W \notin R^3$.

(ii) Let $\underline{u} = (u_1, u_1 + u_3, u_3)$ and $\underline{v} = (v1, v1 + v3, v3)$ be vectors $\in \underline{W}$ and let C number then $\underline{u} + \underline{v} = (u1 + v1, u1 + u3 + v1 + v3, u3 + v3)$

$= (u1 + v1, u1 + v1 + u3 + v3, u3 + v3)$

$= (x1, x1 + x3, x3)$

Where

$\underline{x}1 = u1 + v1$ and $x3 = u3 + v3$

Hence $u + v \in \underline{W}$

Now $c\underline{U} = (cu1, c(u1 + u3), cu3)$

$= (cu1, cu1 + cu3, cu3)$

$= (x1, x1 + x3, x3)$

Where $\underline{x}1 = cu1$ and $\underline{x}\,3 = cu3$

Hence $c\underline{U} \in \underline{W}$

Since $\underline{W}$ is closed under addition and scalar multiplication, then $\underline{W}$ is a subspace of R^3.

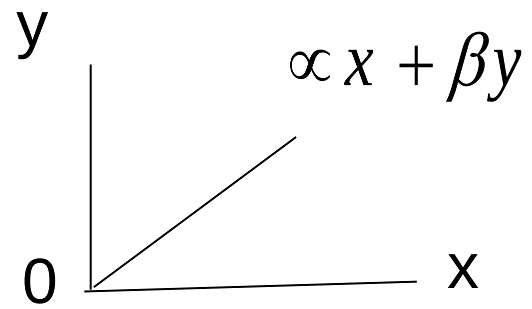
4.3) Spanning sets and linear independence:

Definition:

Let x, y be two vectors is 3- spaces in different directions.

Any vector of the form lies in plane determined by x and y.

Conversely:



Every vector through O in the plane of x, y can be written in the form $\propto \underline{x} + \beta \underline{y}$

We call $\propto \underline{x} + \beta \underline{y}$ a liner combination of x and y.

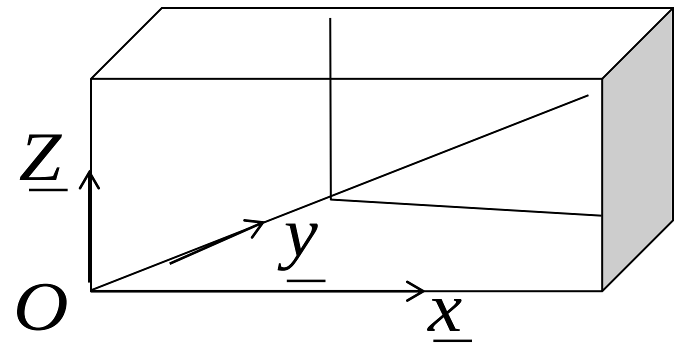
Let x, y and Z be three vectors in 3- space not in the same plane.

Than every vector O, can be written in the form

$\propto \underline{x} + \beta \underline{y} + \propto$ $\underline{z}$ for suitable scalars $\propto, \beta \,\&\, \propto$

This expression is called a linear combination of $\underline{x}$, $\underline{y}$ and $\underline{z}$.

We can extend this definition to Rn



Example: In R4

, 3,10) is a linear combination at (1, 0, 0, 0), (0,1,0,0), (0,0,1,0)

Because

(2,3,1,0) = 2(1,0,0,0) + 3(0,1,0,0) + 3(0,1,0,0) + 1 (0,0,1,0) +

0(0,0,0,0,0)

Example:

Write (3,-1,4,-6) as a linear combination of vector (1,0,3,-1), (2,1,-1,1) and (-1,0,1,1).

Solution:

$(3,-1,4,-6) = \alpha\,(1,0,3,-1) + \beta\,(2,1,-1,1) + \gamma\,(-1,0,1,1)$

$3 = \alpha + 2\beta - \gamma$ (1)

$-1 = \beta$ (2)

$4 = 3\alpha - \beta + \gamma$ (3)

$-6 = -\alpha + \beta + \gamma$ (4)

$-3 = 3\beta$

$5 = \alpha - \gamma$

$3 = 3\alpha + \gamma$

$8 - 4 = \alpha$

$\alpha = 2$

From (4)

$-6 = -2-1+ \gamma$

$-3 = \gamma$

(3, -1,4,-6) = 2(1,0, 3,-1) + (-1) (2,1,-1,1) + (-3) (-1,01,1)

= (3, -1, 4,-6)

In 3- space let $\underline{x}$, $\underline{y}$. $\underline{z}$ be three vectors in the same plan but in different direction then each is a linear combination the other two.

Example:

Suppose $\underline{y} = \left(\frac{3}{2}, +1, \frac{-2}{3}\right), \quad \underline{Z} = \left(-1, \frac{2}{3}, 2\right),$

Then $$\underline{x} = 2\underline{y} + \frac{3}{2}\underline{Z} = \left(\frac{3}{2}, 3, \frac{7}{3}\right)$$

Also $$\underline{y} = \frac{1}{2}\underline{x} - \frac{3}{4}\underline{Z} = \left(\frac{3}{2}, 1, \frac{-2}{3}\right)$$

$$\underline{z} = \frac{2}{3}\underline{x} - \frac{4}{3}\underline{y} = \left(-1, \frac{2}{3}, 2\right)$$

Also we get $2\underline{x} - 4\underline{y} - 3\underline{z} = \underline{0}$

Here is a non-trivial linear combination of $\underline{x}$, $\underline{y}$, $\underline{z}$ equal zero

Note :

(if $\underline{p}$, $\underline{q}$, γ are any three vectors then

$o\underline{\rho} + o\underline{q} + o\underline{\gamma} = \underline{0}$

We call this trivial linear combination ofp,q and γ

Definition:

Let $\underline{x}1$, $\underline{x}2$, …….. $\underline{x}r$ be n-vectors , if $\exists$ scalars $\alpha_1, \alpha_2, .. , \alpha_r$ not.

all zero

Such that $\alpha 1\, \underline{x}1 + \alpha 2\, \underline{x}2 + .. + \alpha r\, \underline{x}r = \underline{0}$ -

Then $\underline{x}1$, $\underline{x}2$, ... $\underline{x}r$ are linearly dependent or from linearly dependent set.

Example:

$$(1,0,2,1) + 3(2,-2,4,2) + (-2)(5,6,7,8) + (-6)(0,-3,1,-1)$$

$$= (0,0,0,0) = \underline{0}$$

$(1,0,2,1)$, $(2,-2,4,2)$, $(5,6,7,8)$ and $(0,-3,1,-1)$

Are linearly dependent

Example:

The victors $\underline{x}$, $\underline{x}$, $\underline{y}$ are linearly dependent

Because

$$1\underline{x} + (-1)\underline{x} + 0\underline{y} = \underline{0}$$

Theorem :

the 3- vectors $\underline{x}$, $\underline{y}$, $\underline{z}$ are linearly dependent then they are coplanar .

Proof :

Since $\underline{x}$, $\underline{y}$, $\underline{z}$ are linearly dependent , scalars α, β, γ (not all zero) such that

$$\alpha\underline{x} + \beta\underline{y} + \gamma\underline{z} = \underline{0}$$

Suppose without loss of generality that

$\alpha \neq 0$ then

$$\alpha\underline{x} = -\beta\underline{y} - \gamma\underline{z} \qquad \underline{x} = \frac{-\beta}{\alpha}\underline{y} - \frac{\gamma}{\alpha}\underline{z}$$

So $\underline{x}$ is alin. Comb A$\underline{y}$, $\underline{z}$

$\therefore$ $\underline{x}$ lies on the plant A$\underline{y}$ and $\underline{z}$.

: $\underline{x}$, $\underline{y}$, $\underline{z}$ are coplanar , they all lie on the same plane.

Theorem:

If $\underline{x}1 , .. , \underline{x}r$ are linearly dependent then at least one conversely.

Proof:

(a) suppose $\underline{x}1 , .. , \underline{x}r$ linearly dependent then

∍ scalars $\alpha , .. , \alpha r$ not all zero such that

$$\alpha 1\, \underline{x}1 + \ldots + \alpha r\, \underline{x}r = \underline{0}$$

then W.L.O.G $\alpha \neq 0$

$$\alpha 1\underline{x}1 = -\alpha 2\underline{x}2 - \ldots - \alpha r\underline{x}r$$

$$\underline{x}_1 = \frac{-\alpha_2}{\alpha}\underline{x}_2 -, \ldots\ldots\ldots, \frac{-\alpha_r}{\alpha_1}\underline{x}_r$$

(b) Conversely if

$x_1 = -\alpha_2/\alpha_1 x_2 - \dots -\alpha_r/\alpha_1 x_r$

Then

$$1\underline{x}_1 - \beta_2\underline{x}_2 - \dots - \beta_r\underline{x}_r = \underline{0}$$

At least one if the scalars $1, -\beta_2, \dots, -\beta_r$

Is non- zero, and

$\underline{x}_1, \dots, \underline{x}_r$ an linearly dependent

Theorem:

any set if vectors containing $\underline{0}$ is linearly dependent.

Definition:

A set a victors, that is not linearly dependent is linearly independent.

Alternative definition:

The victors $\underline{x}1$, ... , xr are linearly independent

If $\alpha 1\ \underline{x}1 + .. + \alpha r\ \underline{x}r = \underline{0}$ only when all Scalars

$\alpha 1, \ldots, \alpha r$ are zero

$$\alpha 1 = \alpha 2 = \ldots = \alpha r = 0$$

Example: the following sets for lin. dep. or lin. inep.

(i) (1,1,-1) , (2, 1, 3) , (7,5,3) in R3

(ii) (1, 1, 0, 1) , (1, -1, 1, 0) , (1, -1, -1, -1) in $\Re^4$

Solution:

(i) Can we find scalars α, β, δ not all zero such that α (1, 1, -1) + β (2,1, 3) + (7, 5, 3) = (0, 0, 0) = <u>0</u>

$$\left.\begin{aligned} \alpha + 2\beta + 7\gamma &= 0 \\ \alpha + \beta + 5\gamma &= 0 \\ -\alpha + 3\beta + 3\gamma &= 0 \end{aligned}\right\} \quad \left.\begin{aligned} \beta + 2\gamma &= 0 \\ 5\beta + 10\gamma &= 0 \end{aligned}\right\} \quad 4\beta + 8\gamma = 0$$

$$\beta = -2\gamma$$

Try $\beta = 2, \qquad \gamma = -1 \qquad \therefore \alpha = 3$

$$\therefore (1,1,-1), \quad (2,1,3), \qquad (7,5,3)$$

$\therefore$ (1, 1, -1) , (2, 1, 3) , (7, 5, 3) are lin. dep.

- Can we find scalars α, β, γ not all zero such that

$$\alpha(1,1,0,1)+\beta(1,-1,1,0)+\gamma(1,-1,-1,-1)=o$$

$$\left.\begin{aligned} \alpha+\beta+\gamma&=o \\ \alpha-\beta-\gamma&=o \\ \beta-\gamma&=o \\ \alpha\qquad-\gamma&=o \end{aligned}\right\}$$

$$\alpha=o$$
$$\therefore \beta=\gamma$$
$$\therefore \alpha=\gamma$$

$$\therefore \alpha=\beta=\gamma=o$$

$\therefore$ (1,1,0,1),(1,-1,1,0),(1,-1,-1,-1)

- Are line ind.p.

Theorem:

(a) Let $\underline{x}_1, \dots, \underline{x}_s \in \Re^n$.

then the set of all linear COMBINATION of $\underline{x}_1, \dots, \underline{x}_s$

Is a subspace S of $\Re^n$.

(b) if T is any subspace of $\Re^n$ containing $\underline{x}_1, \dots, \underline{x}_s$

Then $S \subseteq \mathrm{T}$ (S contain in T).

Proof:

(a) The sum of two linear combination $\underline{x}_1,\cdots,\underline{x}_s$ is another line comb. if $\underline{x}_1,\cdots,\underline{x}_s$ .S satisfies condition (i) for being subspace .

$$(\alpha_1\underline{x}_1+\ldots+\alpha_s\underline{x}_s)+(\beta_1\underline{x}_1+\ldots+\beta_s\underline{x}_s)$$
$$=(\alpha_1+\beta_1)\underline{x}_1+\ldots+(\alpha_s+\beta_s)\underline{x}_s$$

Also $\alpha(\alpha_1\underline{x}_1+\ldots+\alpha_s\underline{x}_s)=(\alpha\alpha_1)\underline{x}_1+\ldots+(\alpha\alpha_s)\underline{x}_s.$

$\therefore$S satisfies condition (ii)

$\therefore$ S is subspace.

(b) if T contains $\underline{x}_1, \cdots, \underline{x}_s$ then T contains all their line. Comb.

$\therefore \quad S \subseteq \mathbf{T}$ (T contained in S).

[**Theorem :** If $\underline{x}_1, \ldots, \underline{x}_r \in$ a subspace S of $\Re^n$

Then every line . Comb .of $\underline{x}_1, \ldots, \underline{x}_r \in$ S.]

Definition:

let S= $\{v_1, v_2, \ldots, v_s\}$ be a subspace of a vector space V. then S is a spanning set of V if every vector in V can be written as a line. Comb of vectors in S .then we say S spans V.

Example:

The set S={(1,0,0),(0,1,0),(0,0,1)} spans

R^3.

Since any vector u = $(u_1, u_2, u_3) \in R^3$

can be written as $u = u_1(1,0,0) + u_2(0,1,0) + u_3(0,0,1)$

$= (u_1, u_2, u_3)$

.

Example:

Let $\underline{x}_1 = (1,0,1,0)$, $\underline{x}_2 = (1,1,0,0)$, $\underline{x}_3 = (0,1,1,1)$

The subspace S spanned by $\underline{x}_1, \underline{x}_2, \underline{x}_3$

consists all vector S of the form :

$$\alpha_1(1,0,1,0) + \alpha_2(1,1,0,0) + \alpha_3(0,1,1,1)$$

$$= (\alpha_1 + \alpha_2, \alpha_2 + \alpha_3, \alpha_1 + \alpha_3, \alpha_3)$$

Theorem :

If We have a spanning set for a subspace S then any set obtaining by adjoining extra element of S is also a spanning set

Sketch of proof :

$$\alpha_1 \underline{x}_1 + \ldots\ldots + \alpha_s \underline{x}_s + \ldots\ldots + O\underline{y}_1 + \ldots\ldots + O\underline{y}_t$$

Theorem :

If $\underline{x}_1, \ldots\ldots, \underline{x}_s$ are line independence and subspace S then no proper sub set of $\underline{x}_1, \ldots\ldots, \underline{x}_s$ can span S.

Sketch of proof :

$x_2,.......,x_s$ for example can not span S. because $\underline{x}_1 \in S$

, but n is not a line. Comb of $\underline{x}_2,.......,\underline{x}_s$

Theorem :

If a set of vector $\{\underline{x}_1,.......,\underline{x}_r\}$ is line dep . then any larger set

$\{\underline{x}_1,....,\underline{x}_r,\underline{y}_1,....,\underline{y}_s\}$ Is line dep.

Proof :

$\exists$ Scalar $\alpha_1,.......,\alpha_r$ not all zero such that

$\alpha_1 n_1 + ... + \alpha_r n_r = \underline{0}$

$\therefore\ \alpha_1\underline{x}_1 + ... + \alpha_r\underline{x}_r + ... + O\underline{y}_1 + ... + O\underline{y}_s = \underline{0}$

$\therefore$ Not all scalars on L.H.S are zero $\underline{x}_1,....,\underline{x}_r,\underline{y}_1,....,\underline{y}_s$

are line dep .

Corollary:

If a set is line indep then any subset is line indep .

4.4) Basis and Dimension :

DEFINITION :Let S be a subspace of R^n. Any line indep set in S that spans S is a basis of S .

Theorem :

If $\underline{x}_1,......,\underline{x}_r$ are line indep vectors in subspace S and if $\underline{y}_1,.......,\underline{y}_s$ span S then $r \leq s$

Theorem :

If $e_1,........,e_m$ and $f_1,......., f_n$ are bases of subspaces S them m=n.

Proof :

$e_1,........,e_m$ are line indep.

$f_1,......., f_n$ span S.

$\therefore m \leq n$

$f_1,......., f_n$ are line indep.

$e_1,........,e_m$ span S.

$\therefore n \leq m$

$\therefore m = n$

We call this number m the dimension of S (dim S).

Example:

Show that S is a basis for R^3 where

S={(1,0,0),(0,1,0),(0,0,1)}.

Solution:

To prove that S is line indep .

$$\alpha_1(1,0,1,0)+\alpha_2(1,1,0,0)+\alpha_3(0,1,1,1)=(0,0,0)$$

$$\therefore \alpha_1 = 0 \qquad \alpha_2 = 0 \qquad \alpha_3 = 0$$

$$\alpha_1 = \alpha_2 = \alpha_3 = 0$$

$\therefore S$ is line indep.

Does S span R^3

Let $u = (u_1, u_2, u_3) \in R^3$

Then $u = u_1(1,0,0) + u_2(0,1,0) + u_3(0,0,1)$

$= (u_1, u_2, u_3)$

S span R^3

$\therefore$ S is a basis for R^3

Example :

Show that S={(1,1),(1,-1)} is a basis for R^2

Solution :

Let $x \in R^2$ where $\underline{x} = (x_1, x_2)$.

To show that $\underline{x}$ can be written as a linear combination of

v_1 and v_2

$$\underline{x} = (x_1, x_2) = c_1 v_1 + c_2 v_2$$
$$= c_1(1,1) + c_2(1,-1)$$
$$= (c_1 + c_2, c_1 - c_2)$$

$$x_1 = c_1 + c_2$$
$$x_2 = c_1 - c_2$$

∴ Since the coefficient matrix of this system has a non-zero determinant then the system has unique solution.

∴ S spans R^2

To show that S is line indep .

$$c_1v_1 + c_2v_2 = 0$$
$$c_1(1,1) + c_2(1,-1) = (0,0)$$
$$(c_1 + c_2, c_1 - c_2) = (0,0)$$
$$c_1 + c_2 = 0$$
$$\underline{c_1 - c_2 = 0}$$
$$2c_1 = 0$$
$$c_1 = c_2 = 0$$

∴ S is line indep

∴ S is a basis for R^2

Theorem:

If $\underline{x}_1,......,\underline{x}_r$ are line indep and if $\underline{x}_{r+1}$ is not a line .comb. of $\underline{x}_1,......,\underline{x}_r$ then $\underline{x}_1,......,\underline{x}_r,\underline{x}_{r+1}$ are line indep.

Proof :

Can we find scalars $\alpha_1,......,\alpha_r,\alpha_{r+1}$ not all zero such that

$$\alpha_1\underline{x}_1+......+\alpha_r\underline{x}_r+\alpha_{r+1}\underline{x}_{r+1}=0 \quad ? \quad .....(1)$$

If equation (1) satisfied then $\alpha_{r+1}=0$ (for other wise $\underline{x}_{r+1}=\frac{-\alpha_1}{\alpha_{r+1}}\underline{x}_1-......-\frac{\alpha_r}{\alpha_{r+1}}\underline{x}_r$)

Hence $\alpha_1\underline{x}_1+......+\alpha_r\underline{x}_r=0$

Hence $\alpha_1=......=\alpha_r=0$ (for other wise $\underline{x}_1,......,\underline{x}_r$ would be line . dep).

$\therefore \underline{x}_1,......,\underline{x}_r,\underline{x}_{r+1}$ are line . Indep.

R^n has a basis :

$$e_1=(1,0,...,0)$$
$$e_2=(0,1,...,0)$$
$$e_3=(0,0,1,...,0)$$
$$\vdots$$
$$e_n=(0,0,...,1)$$

Containing n vectors so $\Re^n$

Proof :

If $\alpha_1(1,0,...,0)+....+\alpha_n(0,0,.....,1)=(0,0,....,0)$

Then $(\alpha_1,\alpha_2,....,\alpha_n)=(0,0,...,0)$

$\therefore \alpha_1=\alpha_2=....=\alpha_n=0$

$\therefore$ vectors are line .indep

Also any vector $(\beta_1,.....,\beta_n)$ in R^n can be written

$\beta_1(1,0,...,0)+\beta_2(0,1,...,0)+....+\beta_n(0,0,...,1)$

$\therefore$ vectors span R^n

Theorem :

Every subspace S of R^n has dimension $\leq n$

(subspace {0} has dim 0)

Corollary:

If A,B are subspaces of R^n and if A is contained in B then dim A $\leq$ dim B.

$$(A \subseteq B)$$

Example :

Find the dimension of the subspace W of R^4 spanned by

$$S = \{(-1,\overset{v_1}{2},5,0),(3,0,\overset{v_2}{1},-2),(-5,\overset{v_3}{4},9,2)\}$$

Solution :

W is spanned by S , S is not a basis for W because S is line .dep

v_3 can be written as line comb. Of v_1, v_2

$$v_3 = 2v_1 - v_2$$

$\therefore W$ is spanned by $S_1 = \{v_1, v_2\}$

S_1 is line. Indep

$\therefore$dim W =2.

INTERSECTIONS AND SUMS OF SUBSPACES:

Let Sand T be subspaces of $\Re^n$. Then their intersection denoted by S n T consist of all vectors common to

S and T .

The sum , S+T of the subspaces S and T is defined as

$$\{\underline{x} + \underline{y}; \underline{x} \in S, \underline{y} \in T\}.$$

THEOREM:

If S,T are subspaces in $\Re^n$,then so are S n T and S+T.

PROOF:

(a) Suppose $\underline{x}, \underline{y} \in SnT$ and α is scalar .then

$\underline{x} \in SnT$ $\qquad \therefore \underline{x} \in S and\, \underline{x} \in T$

$\underline{y} \in SnT$ $\qquad \therefore \underline{y} \in S and\, \underline{y} \in T$

$\therefore \underline{x} + \underline{y} \in S$ (since S is a subspace.)

$\underline{x} + \underline{y} \in T$

$\therefore \underline{x} + \underline{y} \in SnT.$

Also

$\alpha \underline{x} \in S, \alpha \underline{x} \in T$

$\therefore \alpha \underline{x} \in SnT.$

.:S n T is a subspace.

(b) Consider two vectors in S+T

$(\underline{x}_1 + \underline{y}_1) and (\underline{x}_2 + \underline{y}_2)$

Where $\underline{x}_1, \underline{x}_2 \in S, \underline{y}_1, \underline{y}_2 \in T$

$$(\underline{x}_1+\underline{y}_1)+(\underline{x}_2+\underline{y}_2)=(\underline{x}_1+\underline{x}_2)+(\underline{y}_1+\underline{y}_2).$$

$$\underline{x}_1+\underline{x}_2\in S,\ \underline{y}_1+\underline{y}_2\in T.$$

$$\therefore(\underline{x}_1+\underline{x}_2)+(\underline{y}_1+\underline{y}_2)\in S+T.$$

Also

$$\alpha(\underline{x}_1+\underline{y}_1)=\alpha\underbrace{\underline{x}_1}_{\in S}+\alpha\underbrace{\underline{y}_1}_{\in T}$$

$$\therefore(\underline{x}_1+\underline{y}_1)\in S+T.$$

.:S+T is a subspace .

THEOREM:

Dim S n T + dim (S+T)=dim S +dim T.

EXAMPLE:

In 3-space let S,T in plane through the origin . Then

Dim S =dim T= 2.

Dim (S n T)=1 , S+T= $\Re^3$

.:1+3= 2+2).

CHAPTER FIVE:

LINEAR TRANSFORMATIONS:

Geometrical transformation in a plane and in 3-spaces

e.g/ rotation about a point.

reflection in a line.

rotation about a line.

magnitication , stretch , shear .

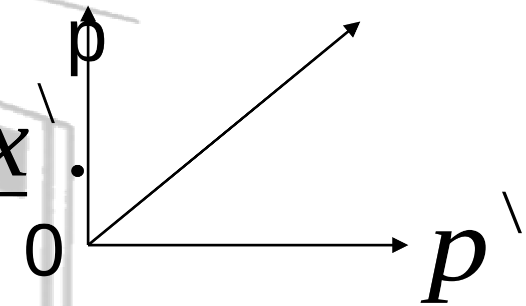
These transformations map points into points .

But if we denote the centre of a rotation by 0 or if we choose a point 0 is the line of a reflection or a rotation then these transformations map vectors with

Origin 0 to a vector with origin 0.

Such a transformation of vectors can be denoted by T and if T transform or map

Vector $\underline{x}\, to\, \underline{x}^{\backslash}$ we write $\underline{x}T = \underline{x}^{\backslash}$.



<u>EX</u>. Shear

Denote shear by S it transform the point

$$(\alpha,\beta)\, to\, (\alpha+\beta,\beta)$$

$$\therefore (\alpha,\beta)S = (\alpha+\beta,\beta).$$

<u>DEFINITION:</u>

If T is any lineear transformation of V into W ,T:V $\longrightarrow$ W then $\forall \underline{x},\underline{y} \in V$ and for any scalar λ

(a) $(\underline{x}+\underline{y})T=\underline{x}T+\underline{y}T.$

(b) $(\lambda\underline{x})T=\lambda(\underline{x}T).$

<u>DEFINITION:</u>

A mapping (or transformation) T from $\Re^m to \Re^n$ is a linear Transformation of

$$(i)(\underline{x}+\underline{y})T=\underline{x}T+\underline{y}T$$

$$(ii)(\lambda\underline{x})T=\lambda(\underline{x}T)$$

For all vectors $\underline{x},\underline{y}\in\Re^n$ and for all scalars $\lambda.$

<u>EXAMPLE:</u>

Show that $T:\Re^2\rightarrow\Re^2 where V=(v_1,v_2)\in\Re^2.$

$$(v_1,v_2)T=(v_1-v_2,v_1+2v_2).$$

SOLUTION:

$$(i)(u+v)T = (u_1+v_1, u_2+v_2)T$$

$$= [(u_1+v_1)-(u_2+v_2), (u_1+v_1)+2(u_2+v_2)]$$

$$= [(u_1-u_2)+(v_1-v_2), (u_1+2u_2)+(v_1+2v_2)]$$

$$= (u_1-u_2, u_1+2u_2)+(v_1-v_2, v_1+2v_2)$$

$$= uT+vT$$

(ii) Since $\lambda u = \lambda(u_1, u_2) = (\lambda u_1, \lambda u_2)$

$$(\lambda u)T = (\lambda u_1, \lambda u_2)T = (\lambda u_1 - \lambda u_2, \lambda u_1 + 2u_2)$$

$$\lambda(u_1 - u_2, u_1 + 2u_2)$$

$= \lambda(uT)$. .:T is a linear transformation.

Properties OF linear transformations •

let T be a linear transformation V into W
when u and v ~~V then~~ the following properties are true :

1) (0)T = 0.
2) (-v)T = -(v)T.
3) (u-v)T = (u)T – (v)T.
4) IF v = $c_1 v_1 + c_2 v_2 + \ldots + c_n v_n$ then

$$(v)T = (c_1 v_1 + c_2 v_2 + \ldots + c_n v_n)T$$

$$= c_1(v_1)T + c_2(v_2)T + \ldots + c_n(v_n)T.$$

EXAMPLE:

Let : $\Re^3 \to \Re^3$ such that

(1,0,0)T = (2,-1,4)

(0,1,0)T = (1,5,-2)

(0,0,1)T =(0,3,1) FIND (2,3,-2)T.

SOLUTION

Since (2,3,-2) = 2(1,0,0) + 3(0,1,0) – 2(0,0,1)

This from property (4)

(2,3,-2)T = 2(1,0,0)T + 3(0,1,0)T – 2(0,0,1)T

=2(2,-1,4) + 3(1,5,-2) – 2(0,3,1)

=(4,-2,8) + (3,15,-6) – (0,6,2)

=(7,7,0)

- <u>DEFINITION:</u>

The set of all image vectors $\underline{X}$ T (when $\underline{x} \in \Re^n$) is called the image of T denoted by imT.

$$\text{imT} = \{ \underline{x}T; x \in \Re^m \}.$$

<u>THEOREM:</u>

IF T is a linear transformation from $\Re^m to \Re^n$ then imT is A subspace of $\Re^n$

<u>PROOF:</u>

Suppose $\underline{u}, \underline{v} \in imT$

then $\underline{u} = \underline{x}T, \underline{v} = \underline{y}T$ where $\underline{x}, \underline{y} \in \Re^m$

$$\therefore \underline{u} + \underline{v} = \underline{x}T + \underline{y}T = (\underline{x} + \underline{y})T \in imT.$$

Also

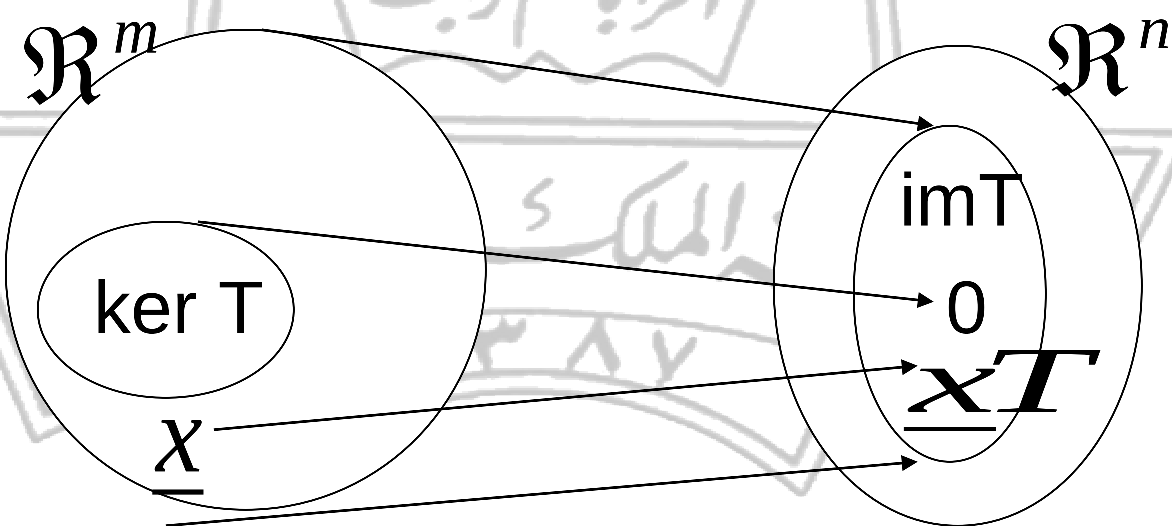
$$\alpha u = \alpha(\underline{x}T) = (\alpha\underline{x})T \in imT.$$

imT satisfies both conditions for being a subspace

DIFINITION:

The set of vectors $\underline{x}$ such that $\underline{x}T = \underline{0}$ is called kernal of T.

Ker T= { $\underline{x}$; $\underline{x}T = \underline{0}$ }.



THEOREM:

The kernel of T: $\Re^m \to \Re^n$ is a sub space of $\Re^m$.

PROOF:

We must prove that if $\underline{x}, \underline{y} \in \ker T$ then

$$\underline{x} + \underline{y} \in \ker T$$

and $\alpha \underline{x} \in \ker T$

THEOREM:

If T is a linear transformation from $\Re^m$ into $\Re^n$ then dim (kerT) + dim (imT) = m .

EXAMPLE:

Find the kernel of the linear transformation

T: $\Re^2 \rightarrow \Re^3$ given by

$$(x_1, x_2)T = (x_1 - 2x_2, 0, -x_1)$$

SOLUTION:

To find ker (T) we need to find all X= (x_1, x_2) such that

$$(x_1, x_2)T = (x_1 - 2x_2, 0, -x_1) = (0,0,0)$$

$$x_1 - 2x_2 = 0$$

$$0 = 0$$

$$-x_1 = 0$$

$$\therefore x_1 = 0$$

$$\therefore x_2 = 0$$

$$(x_1, x_2) = (0,0)$$

.: ker (T) ={(0,0)} ={ $\underline{\mathbf{O}}$ }

EXAMPLE:

Suppose T is a linear transformation from $\Re^4 to \Re^3$ define as follows:

$$(x_1, x_2, x_3, x_4)T = (x_1 + x_3 - x_4, x_2 + x_3 + x_4, x_1 + x_2 + 2x_3)$$

Find the ker (T) .

SOLUTION:

$$\underline{x} \in \ker(T) if\ \underline{x}T = \underline{0}$$

If

$$x_1 + x_3 - x_4 = 0$$
$$x_2 + x_3 + x_4 = 0$$
$$x_1 + x_2 + 2x_3 = 0$$

Put $x_3 = \alpha, x_4 = \beta$ then $x_1 = \beta - \alpha, x_2 = -\alpha - \beta.$

Typical element in ker T is $(-\alpha + \beta, -\alpha - \beta, \alpha, \beta)$

$$= \alpha(-1,-1,1,0) + \beta(1,-1,0,1)$$

Ker(T)=$\{\alpha(-1,-1,1,0).\beta(1,-1,0,1)\}$

The vectors(-1,-1,1,0),(1,-1,0,1) form basis for kerT because they span it and they are line .Ind .

.: dim ker(T) =2

Typical element in image of T

$$(x_1 + x_3 - x_4, x_2 + x_3 + x_4, x_1 + x_2 + 2x_3) = (t_1, t_2, t_3)$$

$$when \quad t_3 = t_1 + t_2$$

So, Typical element in image of T is

$$(t_1, t_2, t_1 + t_2) = t_1(1,0,1) + t_2(0,1,1)$$

.:dim im T=2.

<u>THEOREM:</u>

Let T: V $\longrightarrow$ W be a linear transformation , then T is One-to-one iff ker (T)={_0}.

<u>THEOREM:</u>

Let T: V $\longrightarrow$ W be a linear transformation , then T is Onto iff the rank of T is equal to the dimension of W.